

Scaling law for topologically ordered systems at finite temperature

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(Received 20 November 2008; published 16 April 2009)

Understanding the behavior of topologically ordered lattice systems at finite temperature is a way of assessing their potential as fault-tolerant quantum memories. We compute the natural extension of the topological entanglement entropy for $T > 0$, namely, the subleading correction I_{topo} to the area law for mutual information. Its dependence on T can be written, for Abelian Kitaev models, in terms of information-theoretical functions and readily identifiable scaling behavior, from which the interplay between volume, temperature, and topological order, can be read. These arguments are extended to non-Abelian quantum double models, and numerical results are given for the $D(S_3)$ model, showing qualitative agreement with the Abelian case.

DOI: 10.1103/PhysRevB.79.134303

PACS number(s): 05.30.-d, 11.15.-q, 71.10.Pm

The notion of topological order was first introduced in the context of the fractional quantum Hall effect.¹ It aims at identifying phases that cannot be separated by local-order parameters. Such phases can exhibit exotic phenomena such as topologically protected ground-state degeneracy or quasiparticle excitations, called anyons, with statistical behavior that is different from bosons or fermions.² Besides the fact that they reveal unusual states of matter, topologically ordered systems are also interesting because they might allow for intrinsically fault-tolerant quantum computation.^{3,4} In such systems, the division of a quantum algorithm as initialization, unitary evolution, and readout⁵ would translate into creating anyons, braiding them and fusing them back to the vacuum, respectively. While conceptually appealing, the robustness of this *topological* quantum computation against realistic noise models is far from being fully assessed.

This paper is devoted to quantifying how temperature affects a topologically ordered medium. For that, we will use an entropic topological order parameter I_{topo} and focus on lattice spin systems that are exactly solvable.^{3,6} We will show that, at any fixed temperature, I_{topo} is nonzero only if the size of the system is finite. Importantly, we will exhibit a *scaling* relation describing how a given increase in the system size can be compensated by a vanishing decrease in the temperature. After recalling some notions on the topological entropy, we will provide a complete analysis for the simple toric code model. Then, we will turn to more general models, show how to compute entropic quantities, and provide numerical evidence for scaling laws in the case of the simplest non-Abelian quantum double model, the one based on $D(S_3)$.³ I_{topo} has an information-theoretical meaning in terms of thermal anyonic charge probability distributions and is therefore probably relevant for topological quantum computation,⁴ as will be argued in Ref. 7.

We start our discussion by briefly recalling the notion of topological entropy.^{8,9} It will play a central role in the following. Let us consider a two-dimensional many-body quantum system in a pure state of its ground subspace and let $R:R_c$ denote a bipartition of this system, with R being connected. Let us further assume that our system satisfies an “area” law. That is, the von Neumann entropy of region R

grows linearly with the size of its boundary, $S_R = -\text{tr} \rho_R \ln \rho_R = \alpha' |\partial R| - \gamma' + \delta(R)$, where ρ_R denotes the reduced density matrix corresponding to region R , α' is a constant, and where $\lim_{|\partial R| \rightarrow \infty} \delta(R) = 0$. As first pointed out in Ref. 10 and further analyzed in Refs. 8 and 9 a nonzero value for the constant correction γ' is typical of topologically ordered systems. Indeed, γ' is a topological invariant of the system and is related to the quantum dimension \mathcal{D} of the model at hand as $\gamma' = \ln \mathcal{D}$.

Let us consider a system defined on a torus or on a sphere and divide it into four regions A , B , C , and D (see Fig. 1). We will use

$$I_{\text{topo}} = I_A + I_B + I_C - I_{AB} - I_{AC} - I_{BC} + I_{ABC} \quad (1)$$

as a topological order parameter. ($I_R = S_R + S_{R^c} - S_{R \cup R^c}$ denotes the mutual information between a region R and its complement R^c .) This definition of I_{topo} amounts to replace von Neumann entropies by mutual information in the definition of topological entropy introduced in Ref. 9. At zero temperature, $I_{\text{topo}} = -2\gamma'$. Our choice is motivated by the fact that, at finite temperature, the mutual information between a region R is a measure of the correlations of this region with its environment. The lattice systems we are going to study obey an area law

$$I_R = \alpha |\partial R| - \gamma \quad (2)$$

and have a finite correlation length ξ .¹¹ Therefore, I_{topo} is a quantity where correlations due to a finite ξ cancel out and

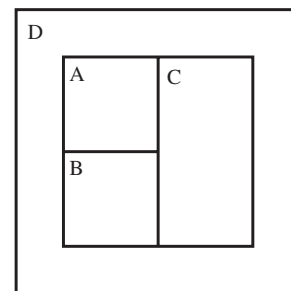


FIG. 1. Division of a torus (or a sphere) into four regions.

that reveals correlations due to topological order only, as the topological entropy γ' at zero temperature.⁸ Direct substitution of Eq. (2) into Eq. (1) shows that $I_{\text{topo}} = \gamma$.¹²

We now want to compute I_{topo} at finite temperature for the toric code.³ Consider a torus T tiled into $L \times L$ square plaquettes. Let us associate a two-level system (“spin”) with each edge of this tiling and let us assume that these spins interact through the Hamiltonian

$$H = - \sum_p B_p - \sum_s A_s, \quad (3)$$

where the index p (s) runs over all plaquettes (vertices) of the tiling. The operators B_p and A_s are defined as $B_p = \prod_{i \in p} \sigma_i^z$ and $A_s = \prod_{i \in s} \sigma_i^x$. The Hamiltonian H is made of local terms, all commuting with each other and, as it turns out, its eigenstates can have arbitrary values for the operators B_p and A_s up to the constraint $\prod_p B_p = \prod_s A_s = 1$. Therefore, these eigenstates are labeled by a triple of quantum numbers: $|\phi, c, w\rangle$. A pattern ϕ denotes the position of all plaquette or “flux-type” excitations, a pattern c indicates the position of all vertex or “charge-type” excitations, and w indexes the topological degeneracy of the state for a fixed configuration of defects. The latter quantum number is made of two bits (w_1 and w_2) that label the values of the integrals of motion of z operators around noncontractible loops on the torus (Wilson loops).³ We have

$$H|\phi, c, w\rangle = (E_0 + 2|\phi| + 2|c|)|\phi, c, w\rangle, \quad (4)$$

where $E_0 = -2L^2$ is the ground-state energy and $|\phi|$ ($|c|$) denotes the number of flux excitations (charge excitations) of the pattern ϕ (c). If P_i denotes the projector onto the sector of energy $E_i = E_0 + 4i$ and $d_i = \text{tr } P_i$ denotes its dimension, we have that $d_i = 4 \sum_{n_\phi, n_c \leq L^2/2} \sum_{n_\phi + n_c = i} \binom{L^2}{2n_\phi} \binom{L^2}{2n_c}$.

We now compute all the thermodynamical quantities we need, assuming that our system is at thermal equilibrium at inverse temperature β . Some details of our calculations will be presented elsewhere.⁷ The partition function reads $Z(\beta, L) = \text{tr } e^{-\beta H} = \sum_{i=0}^{L^2} e^{-\beta E_i} d_i = [(2 \cosh \beta)^{L^2} + (2 \sinh \beta)^{L^2}]^2$ and the von Neumann entropy of the state $\rho = e^{-\beta H} / Z(\beta, L)$ of the whole torus is derived from the identity

$$S_{\text{tot}} = \ln Z - (\beta/Z) \partial Z / \partial \beta. \quad (5)$$

In order to compute the von Neumann entropy of a region R , and eventually I_{topo} , we first observe that $\rho_R(\phi, c, w) = \text{tr}_R |\phi, c, w\rangle \langle \phi, c, w|$ ($R_c = T \setminus R$) depends on w only if R contains noncontractible loops (w cannot be measured locally). Also, up to total charge conservation, $\rho_R(\phi, c, w)$ will only depend on the excitations located on plaquettes and crosses inside R . Therefore, we distinguish between three types of flux excitations: those located on plaquettes with support on R , ϕ_R , those located on plaquettes with support on R_c , ϕ_{R_c} , and those located on plaquettes with support partially on R and partially on R_c , $\phi_{\partial R}$. Similarly, we divide the charge excitations into c_R , c_{R_c} , and $c_{\partial R}$. The crucial observation that helps to compute $\rho_R = \text{tr}_{R_c} e^{-\beta H} / Z(\beta, L)$ and eventually $S(\rho_R)$ is that ∂R fluxes and ∂R charges can be driven inside R_c by application of Pauli operators acting on links of R_c ; that is, $|\phi, c, w\rangle$ is of the form $U_{R_c} |\phi_R, \phi_{R_c}, c_R, c_{R_c}, w\rangle$ for some uni-

tary operator U_{R_c} . To lighten the notations, let us denote by \mathbf{q}_R and \mathbf{q}_{R_c} the patterns of excitations (both flux and charge) in R and in R_c , respectively. All excitations in R (R_c) can be fused into a single excitation q_R^1 ($q_{R_c}^1$) through a unitary operator $U'_R(\mathbf{q}_R, q_R^1) \otimes U'_{R_c}(\mathbf{q}_{R_c}, q_{R_c}^1)$ that relates $|\mathbf{q}_R, \mathbf{q}_{R_c}, w\rangle$ and a state $|q_R^1, q_{R_c}^1, w\rangle$, such that R and R_c contain at most one excited plaquette (vertex). As in Ref. 3, we will call *site* a plaquette and an adjacent vertex. The possibly excited plaquette and the possibly excited vertex within R (R_c) can always be chosen to form a site. The state $|q_R^1, q_{R_c}^1, w\rangle$ can be obtained, from a ground state $|\xi, w\rangle$, by a product of unitary operators along a string connecting the site where q_R^1 is located to the site where $q_{R_c}^1$ is located. In summary, $|\mathbf{q}w\rangle$ can be written as $U_R(\mathbf{q}_R) \otimes U_{R_c}(\mathbf{q}_{R_c}, \mathbf{q}_{\partial R}) |\xi, w\rangle$ for some unitary operators $U_R(\mathbf{q}_R)$ and $U_{R_c}(\mathbf{q}_{R_c}, \mathbf{q}_{\partial R})$. ($\mathbf{q}_{\partial R}$ refers to the charges which are neither fully in R nor fully in R_c .)

We are now in a position to compute S_R . The thermal state of the toric code reads

$$\rho = \sum_{w, \mathbf{q}} \frac{e^{-\beta(E_0 + \Delta E|\mathbf{q}|)}}{Z(\beta, L)} |\mathbf{q}, w\rangle \langle \mathbf{q}, w|, \quad (6)$$

where $\Delta E = 2$ is the energy associated with a single excitation and where $\mathbf{q} = \mathbf{q}_R \cup \mathbf{q}_{R_c} \cup \mathbf{q}_{\partial R}$. Therefore the reduced density matrix of the system R reads $\rho_R = \sum_w \sum_{\mathbf{q}_R, \mathbf{q}_{R_c}, \mathbf{q}_{\partial R}} \frac{e^{-\beta(E_0 + \Delta E|\mathbf{q}|)}}{Z(\beta, L)} \text{tr}_{R_c} [U_R(\mathbf{q}_R) |\xi, w\rangle \langle \xi, w| U_{R_c}(\mathbf{q}_{R_c}, \mathbf{q}_{\partial R})^\dagger]$. To simplify further this expression, we observe that two reduced states $\rho_R(\phi, c, w)$ and $\rho_R(\phi', c', w')$ are orthogonal whenever $(\phi_R, c_R) \neq (\phi'_R, c'_R)$. Indeed two such states can be perfectly distinguished through a measurement of A_s or B_p operators having support on R . Therefore, the sum $\sum_{\mathbf{q}_R}$ is actually a direct sum. If we assume that R is contractible in neither of both directions on the torus, then the values of the Wilson loops w can be revealed by measurements having support fully on R and the sum \sum_w also turns to be a direct sum. So, ρ_R can be written as

$$\rho_R = \bigoplus_{w, \mathbf{q}_R} C(\mathbf{q}_R) \text{tr}_{R_c} U_{R_c}(\mathbf{q}_{R_c}, x) |\xi, w\rangle \langle \xi, w| U_R^\dagger(\mathbf{q}_R, x), \quad (7)$$

where $4C(\mathbf{q}_R) = \frac{4e^{-2\beta|\mathbf{q}_R|}}{Z(\beta, L)} \sum_{\mathbf{q}_{\partial R}, \mathbf{q}_{R_c}} e^{-\beta(E_0 + 2|\mathbf{q}_{\partial R}| + 2|\mathbf{q}_{R_c}|)}$ is the marginal probability of a configuration of defects \mathbf{q}_R . It is decomposition (7) that allows us to compute S_R exactly. From it, we find that S_R separates into a ground-state area contribution and a finite β contribution

$$S_R = S_R^{\text{gs}} + V(\beta, N_p(R), L) + V(\beta, N_*(R), L), \quad (8)$$

where $S_R^{\text{gs}} = S(\text{tr}_R |\xi, w\rangle \langle \xi, w|) = (|\partial R| - 1) \ln 2$ (Ref. 10) and where $N_p(R)$ ($N_*(R)$) denotes the number of plaquettes (crosses) fully contained in R . The function V can be computed exactly using elementary combinatorial identities.⁷ When ρ_R is fully contractible, the direct sum over w appearing in Eq. (7) should be replaced with a simple sum. As a result, the expression for S_R picks a $-\ln 4$ additive correction.

We have used Eq. (8) to compute I_{topo} . We have found that at any finite β , I_{topo} vanishes in the limit where $L \rightarrow \infty$, a result that is consistent with those of Ref. 13 and indicates that (i) the toric code exhibits no temperature-driven phase

transition for I_{topo} and that (ii) the toric code is likely not a scalable quantum memory in the strictest sense⁵ as far as temperature-induced errors are considered.¹⁴ Our computations also reveal that the mutual information between a region R and its complement satisfies an area law of the form Eq. (2). Computing I_{topo} from it in the limit $L \rightarrow \infty$ and considering $|\partial R| = 4\nu L$, with $\nu < 1$, we have found a remarkably simple expression

$$I_{\text{topo}} = 2 \ln 2 - h_2(p_p^{\text{even}}) - h_2(p_*^{\text{even}}), \quad (9)$$

where $h_2(x) = -x \ln(x) - (1-x) \ln(1-x)$ denotes the Shannon entropy of a binary outcome random variables, $p_p^{\text{even}} \simeq p_*^{\text{even}} \simeq (1 + \theta^{\nu^2 L^2})(1 + \theta^{(1-\nu^2)L^2})/2(1 + \theta^{L^2})$ is the probability that region R contains an even number of excited plaquettes (crosses), with $\theta = \tanh \beta$. Equation (9) allows us to understand simply why I_{topo} vanishes at finite temperature when $L \rightarrow \infty$; in this limit, $p_p^{\text{even}} \simeq p_*^{\text{even}} \simeq 1/2$. This equation also reveals a scaling law: at fixed value of ν , I_{topo} only depends on β and L through the parameter $t = \theta^{L^2}$. In particular, a fixed value of t (and thus a fixed value of the topological mutual information) corresponds to the following relations between size and temperature:

$$\beta(t, L) = \ln L - \frac{1}{2} \ln \left(\frac{1}{2} \ln \frac{1}{t} \right) + O(L^{-2}), \quad (10)$$

$$\frac{\partial T(t, L)}{\partial L} = \frac{-1}{L \left[\ln L - \frac{1}{2} \ln \left(\frac{1}{2} \ln \frac{1}{t} \right) + O(L^{-2}) \right]^2} + O(L^{-3}). \quad (11)$$

This last relation tells us how an increase in the size of the system should be compensated by a decrease in temperature in order to maintain a fixed value of the topological entropy. We understand it as an important nuance over the fact that I_{topo} vanishes when $L \rightarrow \infty$ at finite temperature. In particular, it shows that the *rate* at which the temperature should be decreased, in order to maintain a fixed value of γ , *decreases* with the size of the system.

We now turn to a family of non-Abelian models, those based on a quantum double. From now on, we will consider systems defined on an oriented lattice with the geometry of a sphere. A quantum degree of freedom with basis states labeled by the elements of a finite group G is associated with each link of this lattice. These links interact through a Hamiltonian of the form (3), with vertex operators A_s and plaquette operators B_p that still commute. One could write down their form explicitly,³ but it will not be necessary here. A natural way to deal with such models would be to start by providing a description of the complete set of eigenstates of H similar to the one we have used for the toric code. However, when G is non-Abelian, diagonalizing H seems to be a difficult problem. Yet, we can argue that we actually do not need to as far as we are only interested in the topological mutual information. The elementary excitations of H live on sites (a site is a combination of a vertex and an adjacent plaquette). We will restrict to that part of the spectrum of H such that excitations are elementary and pinned at fixed nonadjacent sites. This

restriction can be thought of as additional error correction, where some plaquettes and vertices are overprotected, so that they never get excited by thermal fluctuations (or only with vanishing probability). Therefore, we believe that I_{topo} for this modified model can only be larger than for the full spectrum.

The space of n excitations pinned at fixed sites has the structure³

$$\mathcal{H}[n] = \bigoplus_{q_1, \dots, q_n} \mathcal{H}_{q_1, \dots, q_n}, \quad (12)$$

where each index q_i runs over all possible quasiparticle types for site i . Equation (12) simply means that different excitation patterns lead to orthogonal states. Each space $\mathcal{H}_{q_1, \dots, q_n}$ splits as

$$\mathcal{H}_{q_1, \dots, q_n} = \mathcal{K}_{q_1} \otimes \dots \otimes \mathcal{K}_{q_n} \otimes \mathcal{M}_{q_1, \dots, q_n},$$

where the spaces \mathcal{K}_{q_i} correspond to the local degrees of freedom of the quasiparticles.³ The fusion space $\mathcal{M}_{q_1, \dots, q_n}$ is what makes non-Abelian anyonic systems so special. For Abelian models, for which the fusion rules are trivial, the dimension of the fusion space is equal to 1. A nontrivial fusion space appears when fusing two anyons can yield different quasiparticles,⁵

$$q_a \times q_b = \sum_c N_{ab}^c q_c. \quad (13)$$

The system has to fulfill some neutrality conditions, i.e., its state should be such that fusing all the particles yields the trivial particle denoted¹ with certainty.¹⁵ The dimension of $\mathcal{M}_{q_1, \dots, q_n}$ depends on the tensor N_{**}^* as follows:⁵ $\dim \mathcal{M}_{q_1, \dots, q_n} = \sum_{b_1, \dots, b_{n-2}} N_{q_1 q_2}^{b_1} N_{b_1 q_3}^{b_2} \dots N_{b_{n-2} q_n}^{b_{n-1}}$. We observe that computing this quantity amounts to contracting a (quasi)translationally invariant matrix product state.¹⁶ Likewise $\dim \mathcal{H}[n]$ can be computed efficiently, as well as the partition function of the model,

$$Z(\beta, n) = \sum_{q_1} \dots \sum_{q_n} \prod_{i=1}^n d(q_i) \dim \mathcal{M}_{q_1, \dots, q_n} e^{-\beta E(q_1, \dots, q_n)},$$

where $E(q_1, \dots, q_n)$ is the energy associated with a defect configuration q_1, \dots, q_n and where $d(q_i)$ is the dimension of the space \mathcal{K}_{q_i} .

To compute I_{topo} , we consider a situation where a pair of anyons $|q\bar{q}\rangle$ is created in such a way that anyon q lies in some region R and its antiparticle \bar{q} lies in the complementary region. The von Neumann entropy of region R then reads⁹

$$S(\rho_R) = S(\rho_R^{\text{gs}}) + \ln d_q. \quad (14)$$

The entropy of a region when the system is in a thermal state can be computed once we are able to calculate the entropy of a region when the system is an arbitrary defect configuration. In turn, just as for the toric code, the latter entropy reduces to computing the entropy when there are only *two* anyons in the system and one lies inside the region we are interested in. Let n_R and n_{R_c} denote the number of sites contained in regions R and R_c , respectively ($n_R + n_{R_c} \equiv n$). Let

$\mathbf{q}_R = q_1, \dots, q_{n_R}$ label the configurations of types of anyons living on the sites contained in region R . The total fusion space splits as

$$\mathcal{M}_{\mathbf{q}_R, \mathbf{q}'_{R_c}} = \bigoplus_b \mathcal{M}_{\mathbf{q}_R}^b \otimes \mathcal{M}_{\mathbf{q}'_{R_c}}^{\bar{b}} \otimes \mathcal{M}_{b, \bar{b}}^1, \quad (15)$$

where $\mathcal{M}_{\mathbf{q}_R}^b$ denotes the fusion space associated with event where all \mathbf{q}_R quasiparticles fuse to an anyon of type b .

Due to total anyonic charge conservation, $\dim \mathcal{M}_{b, \bar{b}}^1 = 1$. Decomposition (15) induces the following representation of the thermal state of H :

$$\rho_{\text{th}} = \bigoplus_{\mathbf{q}_R, \mathbf{q}'_{R_c}, b, \mu_1, \mu_2, s_R, s_{R_c}} \frac{e^{-\beta[E(\mathbf{q}_R) + E(\mathbf{q}'_{R_c})]}}{Z(\beta, n)},$$

$$\mathcal{P}(\mathbf{q}_R \rightarrow b, \mu_1, s_R; \mathbf{q}'_{R_c} \rightarrow \bar{b}, \mu_2, s_{R_c}), \quad (16)$$

where the projector $\mathcal{P}(\mathbf{q}_R \rightarrow b, \mu_1, s_R; \mathbf{q}'_{R_c} \rightarrow \bar{b}, \mu_2, s_{R_c})$ refers to a pure state where all \mathbf{q}_R (\mathbf{q}'_{R_c}) quasiparticles contained in R (R_c) fuse to b (\bar{b}) through the channel $\mu_1, 1 \leq \mu_1 \leq \dim \mathcal{M}_{\mathbf{q}_R}^b$ ($\mu_2, 1 \leq \mu_2 \leq \dim \mathcal{M}_{\mathbf{q}'_{R_c}}^{\bar{b}}$). $E(\mathbf{q}_R) = \sum_{j=1}^{n_R} E(q_j)$ denotes the energy associated with the configuration \mathbf{q}_R and s_R is a collective index for the internal degrees of freedom of the quasiparticles contained in region R ; $1 \leq s_R \leq d(\mathbf{q}_R) = \prod_{j=1}^{n_R} d(q_j)$. $E(\mathbf{q}'_{R_c})$ and s_{R_c} are defined likewise for the region R_c . Note that the ground-state energy is now made equal to zero, upon shifting the Hamiltonian by a multiple of the identity.³

The von Neumann entropy of the reduced state of the region R now reads

$$S(\rho_R) = \sum_b \left\{ \frac{Z_R(\beta, b) Z_{R_c}(\beta, \bar{b})}{Z(\beta, n)} [\ln d_b - \ln Z_{R_c}(\beta, \bar{b})] \right\}$$

$$+ \sum_b \frac{Y_R(\beta, b) Z_{R_c}(\beta, \bar{b})}{Z(\beta, n)} + \ln Z(\beta, n), \quad (17)$$

where $Z_R(\beta, b) = \sum_{\mathbf{q}_R} d(\mathbf{q}_R) e^{-\beta E(\mathbf{q}_R)} \dim \mathcal{M}_{\mathbf{q}_R}^b$ and where $Y_R(\beta, b) = \beta \frac{\partial}{\partial \beta} Z_R(\beta, b)$. The von Neumann entropy of the whole sphere is given by Eq. (5), while the ground-state entropy reads⁷ $S_R^{\text{gs}} = \ln |G| (|\partial R| - 1)$.

Equation (17) has allowed us to study numerically how the topological mutual information behaves as a function of β for $G = S_3$, the smallest non-Abelian finite group. Our results are shown in Fig. 2. The systems we have considered are four tiled spheres, all with 96×96 plaquettes. The first sphere contains $n = 64$ sites, the second 144, the third 256, and the fourth 576. Although these systems are small, they are large enough to show that non-Abelian systems are affected by temperature in the same way as the toric code. (i) For a fixed number of sites, there exists a finite width topo-

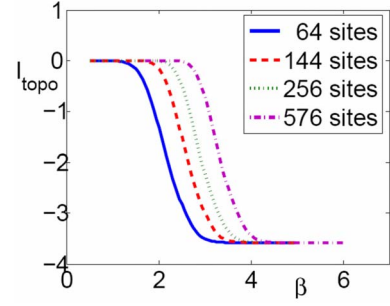


FIG. 2. (Color online) Topological quantum mutual information as a function of the inverse temperature β for the $D(S_3)$ model.

logical phase. (ii) I_{topo} tends to zero for small values of β , and to $-3.58352 = -2 \ln 6$ for sufficiently large values of β , as expected since the order of S_3 is 6. (iii) The larger the number of sites, the larger the value of β where the topological mutual information vanishes. Finally, we have observed that when I_{topo} is plotted as a function of $ne^{-2\beta}$, the curves collapse when n becomes large. We take this observation as strong evidence that the behavior of I_{topo} for the $D(S_3)$ model is governed by the *same* scaling variable as the toric code. Indeed, $\ln(\theta^{L^2}) \approx L^2 e^{-2\beta}$ in the limit where β and L are large ($n \propto L^2$ for the toric code). We therefore believe that the discussion held after Eq. (9) also holds in this case and more generally for any quantum double model.

In conclusion, we have shown that the interplay between thermal effects, lattice size, and topological order (as measured in the subleading correction to the area law) is encoded in well-defined scaling relations (10) and (11). In particular, the rate with which the temperature should be decreased to fight the effect of thermal fluctuations vanishes in the limit of large lattices. These relations seem to hold for both Abelian and non-Abelian systems. As a byproduct, we have derived a formula for the entropy of a region for non-Abelian quantum double models defined on a lattice. This formula depends on the model only through the fusion tensor N_{**}^* and through the energy associated with each quasiparticle. It is therefore tempting to believe that it holds in a more general context. Finally, it is appealing to try to give an operational meaning to I_{topo} by connecting its value with the possibility of using a topologically ordered medium as a robust quantum memory. But to the best of our knowledge, such a connection is still an open problem, even at zero temperature. Equation (9) might contribute to establishing it.

It is a pleasure to thank G. K. Brennen, P. Calabrese, J. I. Cirac, J. I. Latorre, M. A. Martin-Delgado, and A. Stern for fruitful discussions. Financial support from the Spanish Ministry of Science, Comunicad de Madrid and the Generalitat de Catalunya, MEC (Spain), QAP (EU), and Spanish Grants No. MTM2005-00082 ,No. CCG07-UCM/ESP-2797, and I-MATH are acknowledged.

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